Speculative Inflations under Currency Backing

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Abstract

This paper studies the possibility of speculative inflations when a government provides real backing for its currency. When such a backing is costless, speculative inflations are impossible as Obstfeld and Rogoff (1983) have concluded earlier. However, if backing involves resource costs, speculative inflations may emerge as a result of the backing policy.

I. Introduction

That there exist speculative hyperinflationary equilibria in dynamic rational expectations monetary models is now well known.[1] Scheinkman (1980) has concluded that their existence has to do with the possibility that an economy operates in the absence of fiat money. Two strands of research have followed. The first is to impose restrictions on preferences and/or technology so that an economy cannot survive without the use of fiat money. Such conditions have been verified by, among others, Brock and Scheinkman (1980), Scheinkman (1980) and Gray (1984). However, as Obstfeld and Rogoff (1983) have pointed out, those restrictions were economically unreasonable. Pursuing the matter from a policy perspective, they have shown that, when a government backs its currency by guaranteeing an arbitrarily small real redemption value, speculative hyperinflations are impossible; moreover, this result holds even if agents are not completely certain that they can redeem their money in any given period.

The Obstfeld-Rogoff results have been cited extensively in the literature.[2] The policy implications are staggering: For, they have suggested that "the backing schemes . . . preclude speculative hyperinflations even though the government need never actually execute the exchange it offers to make" (Obstfeld and Rogoff 1983, p. 685); and in a different occasion, "the fractional backing . . . is an example of an official guarantee which, though never exercised, precludes inefficient equilibria supported by self-fulfilling expectations" (Obstfeld 1986, p.79).

These implications are quite doubtful. Speculative hyperinflations may emerge when a government *never* redeems its currency. How are they elim-

[1] See, e.g., Wallace (1980) and the references cited therein.
[2] See, e.g., Tirole (1985), Brock (1988), Blanchard and Fischer (1989).

inated when a government *never actually* redeems its currency? Or, put it in a more general perspective, does a government's inaction forestall speculations or precipitate them?

In this paper, we propose a resolution that is born out of an explicit consideration of collection costs of reserves used for currency backing. It is shown that the existence of resource costs of backing a currency and how a currency is backed could affect the types of equilibria that may arise. We give examples in which a government's action as well as inaction both could induce speculations.

This paper proceeds as follows. The basic model is described in the next section. In section III, we develop backing schemes with resource costs modeled explicitly. In section IV, we conclude the paper and discuss possible extensions.

II. The Model

The economy consists of a representative individual. The agent lives forever and maximizes the present discounted value of his utility stream,

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) + v(m_t)] \qquad 1 > \beta > 0, \tag{1}$$

where c_t denotes the representative individual's consumption at time t, m_t is the ratio of nominal money holdings M_t to the price level P_t, and β is the subjective discount factor. In the formulation, $u(\cdot)$ and $v(\cdot)$ are strictly concave, strictly increasing, continuously differentiable utility functions on the open interval $(0, \infty)$, and obey the Inada conditions: $\lim_{c \to 0} u'(c) = \infty, \lim_{c \to \infty} u'(c) = 0, \lim_{m \to 0} v'(m) = \infty, \lim_{m \to \infty} v'(m) = 0$. Also, in order to ensure the viability of hyperinflation equilibria, we impose the following survival condition:[3]

$$\lim_{m\to 0} v(m) > -\infty. \tag{$\star$}$$

The individual holds a single type of assets: fiat money. The budget constraint is

$$M_t = P_t y + M_{t-1} - P_t c_t, \tag{2}$$

where M_t denotes nominal money holding and c_t denotes consumption in period t; y is the constant non-storable endowment in each period; and M_{-1} is given.

Maximizing (1) subject to (2), the first-order (Euler) equation characterizing an optimal path for the individual's problem is

$$\frac{u'(c_t) - v'(m_t)}{P_t} = \beta \frac{u'(c_{t+1})}{P_{t+1}}. \tag{3}$$

Markets clear in each period. There is no government consumption, and the money supply is constant at level M.

In equilibrium, the private demand for consumption during period t, c_t, must equal the aggregate endowment y. The demand for nominal balances, M_t, must equal the supply of fiat money, M, for all t. Multiplying (3) by M and substituting y for c_t and c_{t+1} in (3), we obtain

$$[u'(y) - v'(m_t)]m_t = \beta u'(y) m_{t+1}, \tag{4}$$

A unique positive stationary level of real balances $\bar{m}$ can be solved from (4) with $m_t = \bar{m}$ and $P_t = \bar{P} = M/\bar{m}$ for all t. This is the steady-state Euler path referred to by Obstfeld and Rogoff.

In addition to the steady-state path, there are non-steady-state paths that also satisfy (4). To illustrate, we use the diagrammatic technique

[3] As Obstfeld and Rogoff (1983, fn. 15) have demonstrated that condition ($\star$) is sufficient, but not necessary, for the existence of hyperinflationary equilibria.

employed by Obstfeld and Rogoff. Let $A(m) \equiv [u'(y) - v'(m)]m$ and $B(m) \equiv \beta u'(y)m$. The difference equation (4) can be written as

$$A(m_t) = B(m_{t+1}). \tag{5}$$

By the Theorem of Obstfeld and Rogoff (1983, p.683), the survival condition ($\star$) implies $\lim_{m\to 0} mv'(m) = 0$. Therefore, $A(0) = B(0) = 0$. Figure 1 depicts the dynamics dictated by (5).

($Insert\ Figure$ 1)

Consider the path beginning at $m_0 < \bar{m}$ in figure 1. The initial price level is $P_0 = M/m_0$. Equation (5) implies that the individual's real balance holding converges to zero eventually. Therefore, the sequence of price levels explodes to infinity in a finite number of periods. These fly-away paths are known as speculative hyperinflationary equilibria. In the next section, we investigate government policies that may be used to rule them out. Specifically, we study the possibility of providing reserves to redeem currency.

III. Currency Backing with Resource Costs

Consider a government that stands ready to redeem its currency at some nominal price. We assume that the government initially holds no reserves and imposes a lump-sum tax to raise reserves. The taxation involves a permanent loss of resources equivalent to γ units of goods per period.[4] Finally, we assume that, after the redemption of currency, the government transfers

[4] Alternatively, one may make the assumption that the redemption of currency is financed by real borrowing. Tax revenue must be raised (at a cost) for an indefinite period to service the debt. In either case, the redemption of the currency gives a continuing tax collection cost to the economy. Formally, the resource costs here resemble the monitoring costs as in Townsend (1979) and Williamson (1986).

the currency back to the individual at the end of each period so that the stock of money supply remains at the constant level M.

Because the tax is lump-sum, (3) still represents the Euler equation characterizing the individual's optimizing condition. However, because of backing, the aggregate endowment is reduced to $y - \gamma$. Substituting into (3), and in equilibrium, we obtain

$$[u'(y-\gamma) - v'(m_t)]m_t = \beta u'(y-\gamma)m_{t+1}. \tag{6}$$

A constant level of real balances, m_γ, and prices, $P_\gamma (= M/m_\gamma)$ can be solved from (6) for all t.

When a benevolent government is backing its currency, the backing price may be set equal to or anywhere above P_γ, since setting the it below P_γ will cause a complete withdrawal of fiat money from the market, thus violating (6). Setting it strictly above P_γ is feasible, but may not be desirable as higher price levels imply lower real balances and less utilities. Therefore, we assume that the government sets the backing price to P_γ and the individual is indifferent between trading in the market or exchanging money for goods with the government.[5]

Two points about the backing scheme worth noticing. The first is that the backing price P_γ is strictly above $\bar{P}$ as $\bar{m} > m_\gamma$.[6] The higher price level reflects less resources available under backing relative to the steady-state Euler path in which no backing is involved. The relationship between γ and m_γ can be seen by differentiating (6),

$$\frac{dm_\gamma}{d\gamma} = -\frac{(1-\beta)u''(y-\gamma)m_\gamma}{v''(m_\gamma)} < 0. \tag{7}$$

[5]Technically, there is always the possibility that the private market will shut down and everyone trades with the government at the backing price P_γ. Here we invoke the perfect competition hypothesis that a government is but a small participant in this economy.

[6]To see that, notice $v'(\bar{m}) = (1-\beta)u'(y)$ and $v'(m_\gamma) = (1-\beta)u'(y-\gamma)$.

Condition (7) confirms the intuition that the higher the cost of backing is, the lower is the value of currency. As γ approaches y from below, m_γ converges to zero and P_γ diverges to infinity.

So far, we have examined a deterministic backing scheme in which a constant price level is maintained, but some resources were used up in each period. A benevolent government can maintain the same level as price ceiling and save, at least some, resource costs with a stochastic backing scheme. We will present a version of such a scheme and state the rules of backing as follows:

(i) The government first announces a price ceiling P_γ, beyond which it imposes a lump-sum tax to back currency.

(ii) If the price level is below the ceiling in period t, it is expected that the government chooses to back currency in period $t+1$ depending on the market price level in t. Without loss of generality, we assume that there is a finite set $N = \{1, \cdots, n\}$ such that, if the previous price level is $P^i, i \in N$, people expect that the government backs its currency with probability π_i.

The backing rules are worth some comments. Rule (i) is the deterministic backing rule aforementioned. Other possibilities are examined in the next section. Rule (ii) is a generalization of the stochastic backing rule of Obstfeld and Rogoff. Restricting prices to a finite set may not be sensible as prices should, in principle, take any positive value. This restriction makes sense only if equilibrium prices only take values in a finite set. That is the case as our analysis will be focusing on the (stochastic) stationary equilibria.

Under these backing rules, when an economy begins at P_γ, it remains there forever with aggregate consumption reduced to $y - \gamma$ and price level equal to P_γ.

When an economy begins at P_t^i below P_γ, agents expect that it will be at P_γ with probability π_i, or, at some P_{t+1}^j below P_γ with probability π_{ij}, $i, j \in N$. Therefore, the matrix

$$\Pi = \begin{bmatrix} (\pi_{ij}) & (\pi_i) \\ 0 & 1 \end{bmatrix}$$

represents the transition probability matrix describing the individual's expectation about endogenous price movement as well as government's backing. In equilibrium, this means that, given m_t^i and m_γ, the level of real balances that prevails if the government does not intervene, m_{t+1}^j, satisfies

$$[u'(y) - v'(m_t^i)]m_t^i = \beta\Big[\sum_{j=1}^{n} \pi_{ij} u'(y) m_{t+1}^j + \pi_i u'(y-\gamma) m_\gamma\Big]. \qquad (8)$$

where $i \in N$. Focusing on the stationary solution of (8), time indices can be omitted and (8) becomes

$$[u'(y) - v'(m^i)]m^i = \beta\Big[\sum_{j=1}^{n} \pi_{ij} u'(y) m^j + \pi_i u'(y-\gamma) m_\gamma\Big]. \qquad (9)$$

In the appendix, it is verified that there exists a unique solution $\hat{m} \equiv (\hat{m}^1, \ldots, \hat{m}^n)$ to (9). For $\hat{m}$ to be consistent with the backing rule (*iii*), it is necessary to verify $\hat{m}^i > m_\gamma$, i.e., the price level $\hat{P}^i$ ($\equiv M/\hat{m}^i$) is below the official price ceiling P_γ.

To that end, let us first consider the following equation in m for a given $\pi \in (0, 1)$:

$$[u'(y) - v'(m)]m = \beta[(1-\pi)u'(y)m + \pi u'(y-\gamma)m_\gamma]. \qquad (10)$$

From the previous result, (10) has a unique solution $m(\pi)$. It is easy to verify $m(\pi) > m_\gamma$ and, for m such that

$$[u'(y) - v'(m)]m > \beta[(1-\pi)u'(y)m + \pi u'(y-\gamma)m_\gamma],$$

we have $m > m(\pi)$.[7] Next, let $\hat{m}^s \equiv min_i\, \hat{m}^i$. For s, (9) implies

$$\begin{aligned} & [u'(y) - v'(\hat{m}^s)]\hat{m}^s \\ = \;& \beta\Big[\sum_{j=1}^{n} \pi_{sj} u'(y)\hat{m}^j + \pi_s u'(y-\gamma)m_\gamma\Big] \\ > \;& \beta[(1-\pi_s)u'(y)\hat{m}^s + \pi_s u'(y-\gamma)m_\gamma]. \end{aligned}$$

The last inequality implies $\hat{m}^s > m(\pi_s)$, therefore, $\hat{m}^i > m_\gamma$ and $\hat{P}^i < P_\gamma, i \in N$.

Since the equilibrium is unique for a given Π, we may illustrate the relationship between the existence of speculative inflations and possible backing of a government by examining configurations of Π. This is done in the following through a series of examples.

Case (1): $\pi_i = 0, i \in N$.

In this case, the steady-state Euler path $\hat{m}^i = \bar{m}$ is the only positive solution to (9). This is an economy in which a government never backs its currency. Note that, under current specification of expectation, speculative hyperinflationary paths in figure 1 are also equilibria, except that they are nonstationary. Therefore, one might conclude that government's inaction induces price instability.

Case (2): $\pi_i > 0$, $\pi_{ii} > 0$, and $\pi_{ij} = 0$ for $i \neq j$ and $i, j \in N$.

[7] To demonstrate $m(\pi) > m_\gamma$, it helps to write equations (6) and (10) as follows:

$$\begin{aligned} [(1-\beta(1-\pi))u'(y-\gamma) - v'(m_\gamma)]m_\gamma &= \beta\pi u'(y-\gamma)m_\gamma, \\ [(1-\beta(1-\pi))u'(y) - v'(m(\pi))]m(\pi) &= \beta\pi u'(y-\gamma)m_\gamma. \end{aligned}$$

If $m_\gamma \geq m(\pi)$, we obtain

$$0 = [(1-\beta(1-\pi))u'(y-\gamma) - v'(m_\gamma)]m_\gamma - [(1-\beta(1-\pi))u'(y) - v'(m(\pi))]m(\pi) > 0,$$

which is a contradiction. The second claim can be shown similarly.

In this case, $\hat{m}^i > m_\gamma$. If an economy starts at $\hat{P}^i$, it remains there until a future (random) date that its price level jumps to P_γ once and for all. This is an economy in which the backing policy induces a price escalation. It is noticed that the one-time price-jump path is different from the kind of hyperinflationary equilibria in figure 1. The following case addresses that possibility.

Case (3): $\pi_{ii} = 1 - \pi$, $\pi_{i,i+1} = \pi$, $i = 1, \ldots, n-1$, $\pi_{nn} = 1 - \pi$, $\pi_n = \pi$, and the utility function $v(\cdot)$ satisfies the following restriction: $v'(m) + mv''(m) > 0$ for $m > 0$.[8]

To study the equilibrium behavior in this case, let us first consider the following equation in m,

$$[u'(x) - v'(m)]m = \beta[(1 - \pi)u'(x)m + \pi u'(y - \gamma)m_\gamma]. \tag{11}$$

For $x \in [y - \gamma, y]$, we have demonstrated that (11) has a unique solution m_x. Three properties of (11) are noted: First, $m_y = \hat{m}^n > m_\gamma$. Second, the term $\beta\pi u'(y - \gamma)m_\gamma$ may be viewed as a constant, say, ω, then,

$$\frac{dm_x}{d\omega} = \frac{1}{D},$$

where $D \equiv [u'(x) - v'(m_x)] - \beta(1 - \pi)u'(x) - m_x v''(m_x)$. From (11), $[u'(x) - v'(m_x)] - \beta(1 - \pi)u'(x) = \beta\pi u'(y - \gamma)(m_\gamma/m_x) > 0$. Therefore, $D > 0$ and $dm_x/d\omega > 0$. Third, the change in $B(x, m_x)$, the value of the right-hand-side of (11), due to variations in x can be expressed as follows:

$$\frac{dB}{dx} = -\frac{\beta(1 - \pi)u''(x)m_x[v'(m_x) + m_x v''(m_x)]}{D}.$$

[8] For $i = 1, \ldots, n-1$, the government does not back its currency. An individual may still regard these states being different. This type of uncertainty has been labelled "sunspots" by Azariades (1981) and Cass and Shell (1983).

Since $v'(m) + mv''(m) > 0$, $dB/dx > 0$. Recall the definition of $\hat{m}^n$, this implies

$$\begin{aligned} B(y, \hat{m}^n) &\equiv \beta[(1-\pi)u'(y)\hat{m}^n + \pi u'(y-\gamma)m_\gamma] \\ > B(y-\gamma, m_\gamma) &\equiv \beta[(1-\pi)u'(y-\gamma)m_\gamma + \pi u'(y-\gamma)m_\gamma], \end{aligned}$$

and $u'(y)\hat{m}^n > u'(y-\gamma)m_\gamma$.

Next, given $\hat{m}^n$, let us consider the following equation in m,

$$[u'(y) - v'(m)]m = \beta[(1-\pi)u'(y)m + \pi u'(y)\hat{m}^n].$$

Notice that $\hat{m}^{n-1}$ is the solution to the foregoing equation. Since $u'(y)\hat{m}^n > u'(y-\gamma)m_\gamma$, $\hat{m}^{n-1} > \hat{m}^n$. Following the same procedure, we obtain

$$\hat{m}^1 > \cdots > \hat{m}^n > m_\gamma \quad \text{and} \quad \hat{P}^1 < \cdots < \hat{P}^n < P_\gamma.$$

In this economy the general price level may start at any $\hat{P}^i$, then it rises monotonically to P_γ in a finite number of periods with probability one. The economic story behind this example can be understood as follows: The government first announces a fixed price ceiling P_γ, and an intention to back its currency with probability π only when the price level is sufficiently high, say, $\hat{P}^n$. The public react to such an announcement with expectations such that the government is forced to validate the speculative inflations *ex post*.[9] In this sense, price instability is the result of government's action.

As noted before, P_γ is lower the less costly it is for the government to back its currency. If $\gamma = 0$, $P_\gamma = \bar{P} = \hat{P}^i, \forall i$. The steady-state Euler path is the unique equilibrium. There are no speculative inflations because it does not cost anything for a government to back its currency. On the other hand,

[9] This scenario is similar to the private-sector triggers of government action as discussed in Flood and Garber (1984)

if $\gamma = y$, $P_\gamma = \infty$, Case (3) clearly illustrates the possibility of (stochastic) speculative hyperinflations. They emerge because agents expect so and the government is not able to block their emergence.

IV. Extensions and Conclusions

As we started by critically reviewing the results of Obstfeld and Rogoff. It may be instructive, at this stage, to compare the key difference between the present setting and that of Obstfeld and Rogoff that leads to different conclusions. The answer lies in the Obstfeld-Rogoff setting that the government initially holds claims to capital. These claims of dividends are transfers ordinarily, but can be used as currency reserves if necessary. Effectively, this means that the government's collection costs for reserves are zero.[10] Therefore, no speculative hyperinflations could arise in either deterministic or stochastic backing scheme they designed.

The essence of this paper is, of course, to examine the robustness of that conclusion when currency backing involves resource costs. We have demonstrated that speculative inflations are impossible only if the government is able to costlessly provide backing for its currency. When such backing is costly, a deterministic backing may forestall any speculations, but squander resources. There is no causal statement can be made regarding a stochastic backing policy and the emergence of speculative inflations. That is, speculative inflations may occur when no backing is expected, and they may also occur as a result of the expected backing policy.

Our results hold for two possible extensions. Instead of assuming a

[10]If the government did not transfer the dividends back to consumers, or equivalently, the dividends were put away as "genuine" reserves not available for consumption purposes, the types of stationary equilibria that would arise will be the same as those in the present paper.

permanent costs γ of backing, one may assume a one-time (temporary) costs γ and no other costs will ever incur in the future. This assumption does not change our results in any significant manner. Consider in period T that a government decides to back its currency, this implies that the levels of real balances $(m_T, m_{T+1}, \cdots)$ satisfy

$$[u'(y-\gamma) - v'(m_T)]m_T = \beta u'(y) m_{T+1}, \tag{12}$$

$$[u'(y) - v'(m_s)]m_s = \beta u'(y) m_{s+1}, \quad s \geq T+1. \tag{13}$$

If m_{T+1} is less than the steady-state Euler path $\bar{m}$, equations (13) imply $\lim_{t\to\infty} m_t = 0$ so any finite price ceiling is violated.[11] Therefore, the only positive sequence $\{m_s\}$, $s \geq T+1$, that satisfies equations (13) is $\bar{m}$. This implies a unique $\hat{m}_T$ satisfying (12) and $\hat{m}_T < \bar{m}$. Clearly. our previous analysis regarding the stochastic backing scheme still applies for periods $s < T$.

Second, instead of assuming a permanent backing as represented by the last row of the transition probability matrix Π, one may assume an occasional backing with a general probability distribution. The main effect of this change is that there will be equilibria exhibiting random permanent fluctuations, unlike the present case all possible fluctuations are transient. An example is sufficient to demonstrate this. Consider the stochastic backing rule that a government backs its currency with a probability $\pi \in (0,1)$ that does not depend on what has occurred in the previous period.[12] The transition probability matrix Π now becomes

$$\begin{bmatrix} 1-\pi & \pi \\ 1-\pi & \pi \end{bmatrix}$$

[11] If $m_{T+1} > \bar{m}$, equations (13) imply $\lim_{t\to\infty} m_t = \infty$ and a transversality condition is violated (Obstfeld and Rogoff, 1983)

[12] This is the exact backing rule suggested by Obstfeld and Rogoff 1983, p.685.

implying following equations similar to (9):

$$\begin{aligned} [u'(y) - v'(m^1)]m^1 &= \beta[(1-\pi)u'(y)m^1 + \pi u'(y-\gamma)m^2] \\ [u'(y) - v'(y)]m^2 &= \beta[(1-\pi)u'(y)m^1 + \pi u'(y-\gamma)m^2]. \end{aligned}$$

It is straightforward to show that a unique solution exists such that $\hat{m}^1 > \hat{m}^2 > m_\gamma$. Thus, the backing scheme induces a permanent fluctuation of prices.

Finally, it is noted that we have assumed throughout this paper a constant level of collection costs regardless of the amount of reserves. Two possible variations worth mentioning. One may assume that the amount of reserves, R, is a decreasing function of the collection costs, γ, i.e.,

$$R = R(\gamma), \quad R(0) = \bar{R} > 0, \quad R' \leq 0.$$

For given R, the lowest price at which a government may redeem its currency is $P(R) \equiv M/R$. If $P(\bar{R}) \leq \bar{P}$, the steady-state Euler path could be maintained with an announcement of currency redemption at any prices $P \geq P(\bar{R})$. However, if $P(\bar{R}) > \bar{P}$, our previous analyses remain unaffected. Another variation is to assume that the cost of collection is an increasing function of the amount of reserves, i.e.,

$$\gamma = \gamma(R), \quad \gamma(0) = 0, \quad \gamma' \geq 0.$$

If there exists an $\epsilon > 0$ such that

$$\gamma(R) = 0, \quad R \in [0, \epsilon],$$

the steady-state Euler path may also be maintained provided the price ceiling is announced within the range of $[M/\epsilon, \infty)$. However, if $\gamma(R) > 0$ for all $R > 0$, the speculative inflations may reemerge under the stochastic backing rule.

Appendix

For future references, we first define $m(\phi)$ as the solution to $[u'(y) - v'(m)]m = \phi$ where ϕ is any nonnegative numbers. It is straightforward to show that $m(0) = \tilde{m}$ and $m(\phi)$ is an increasing function of ϕ. Since $\lim_{m\to\infty} v'(m) = 0$, there exists a $\phi^* > 0$ such that

$$m(\phi^*) = \frac{\phi^*}{u'(y) - v'(m(\phi^*))} \leq \frac{\phi^* - \beta u'(y-\gamma)m_\gamma}{(1-\pi_\alpha)\beta u'(y)},$$

where $1 > \pi_\alpha \equiv \min_{i\in N}\ \pi_i \geq 0$.

Finally, define an operator $H \equiv (h^1, ..., h^n)$ mapping from R^n to itself:

$$h^i(z) \equiv \beta \sum_{j=1}^{n} \pi_{ij} u'(y) m(z^j) + \beta \pi_i u'(y-\gamma) m_\gamma,$$

where $z \equiv (z^1, ..., z^n)$. Clearly, H is continuous, $h^i(0) \geq 0$, and

$$\begin{aligned}
h^i(\phi^*) &= \beta \sum_{j=1}^{n} \pi_{ij} u'(y) m(\phi^*) + \beta \pi_i u'(y-\gamma) m_\gamma \\
&\leq \beta \sum_{j=1}^{n} \pi_{ij} u'(y) \frac{\phi^* - \beta u'(y-\gamma)m_\gamma}{(1-\pi_\alpha)\beta u'(y)} + \beta \pi_i u'(y-\gamma) m_\gamma \\
&= \beta(1-\pi_i) u'(y) \frac{\phi^* - \beta u'(y-\gamma)m_\gamma}{(1-\pi_\alpha)\beta u'(y)} + \beta \pi_i u'(y-\gamma) m_\gamma \\
&\leq \phi^* - \beta u'(y-\gamma)m_\gamma + \beta \pi_i u'(y-\gamma) m_\gamma \\
&\leq \phi^*.
\end{aligned}$$

Therefore, H maps a compact, convex set,

$$S \equiv \{(s^1, ..., s^n) : 0 \leq s^i \leq \phi^*\},$$

into a proper subset of itself, a fixed point $\hat{s} \equiv (\hat{s}^1, \ldots, \hat{s}^n)$ exists by the Brower fixed-point theorem, and $\hat{s}$ satisfies

$$\hat{s}^i = \beta[\sum_{j=1}^{n} \pi_{ij} u'(y) m(\hat{s}^j) + \pi_i u'(y-\gamma) m_\gamma].$$

Let $\hat{m}^i \equiv m(\hat{s}^i)$ and $\hat{m} \equiv (\hat{m}^1, ..., \hat{m}^n)$, we have

$$[u'(y) - v'(\hat{m}^i)]\hat{m}^i = \beta\Big[\sum_{j=1}^{n} \pi_{ij} u'(y)\hat{m}^j + \pi_i u'(y-\gamma)m_\gamma\Big].$$

Therefore, $\hat{m}$ is a solution to (9). In the following, we demonstrate that $\hat{m}$ is also unique.

As H is monotone, i.e., for $a \geq b$, $Ha \geq Hb$, there is no loss of generality to focus on S to establish the uniqueness of a solution to (9). To see that, note that the fixed-point relation allows us to define a mapping $K = (k^1, \cdots, k^n)$ from R^n into itself as follows:

$$k^i(m) \equiv \beta\Big[\sum_{j=1}^{n} \pi_{ij} u'(y)m^j + \pi_i u'(y-\gamma)m_\gamma\Big] - [u'(y) - v'(m^i)]m^i,$$

where $m \equiv (m^1, ..., m^n)$. The set of solutions to (9) contains those points $m \in S$ such that $K(m) = 0$ with S defined as before. Since S is a smooth n-manifold with boundary, the number of equilibria in S is finite (Milnor, 1965, p.8). The Poincare-Hopf index theorem (Milnor, 1965, p.20) implies that the sum of the indices of solutions in S is +1. The index of a solution, or zero of $K(m)$, is defined as +1 if the determinant of the negative of the Jacobian matrix of K, $det(-K)$ is positive. If we can show $det(-K) > 0$ at any solution point, the solution must be unique on S.

To this end, consider

$$det(-K) = \begin{vmatrix} \begin{matrix} -\beta\pi_{11}u'(y) - m^1 v''(m^1) \\ +[u'(y) - v'(m^1)] \end{matrix} & \ldots & -\beta\pi_{1n}u'(y) \\ \vdots & \ldots & \vdots \\ -\beta\pi_{n1}u'(y) & \ldots & \begin{matrix} -\beta\pi_{nn}u'(y) - m^n v''(m^n) \\ +[u'(y) - v'(m^n)] \end{matrix} \end{vmatrix}.$$

Multiply $det(-K)$ by $m^1 \cdots m^n$ we obtain

$$\left| \begin{array}{ccc} -\beta\pi_{11}u'(y)m^1 - (m^1)^2 v''(m^1) & \dots & -\beta\pi_{1n}u'(y)m^n \\ +[u'(y) - v'(m^1)]m^1 & & \\ \vdots & \dots & \vdots \\ -\beta\pi_{n1}u'(y)m^1 & \dots & -\beta\pi_{nn}u'(y)m^n - (m^n)^2 v''(m^n) \\ & & +[u'(y) - v'(m^n)]m^n \end{array} \right| .$$

At a fixed point, the diagonal elements of the second matrix,

$$\begin{aligned} & -\beta\pi_{ii}u'(y)\hat{m}^i - (\hat{m}^i)^2 v''(\hat{m}^i) + [u'(y) - v'(\hat{m}^i)]\hat{m}^i \\ = & -\beta\pi_{ii}u'(y)\hat{m}^i - (\hat{m}^i)^2 v''(\hat{m}^i) + \beta \sum_{j=1}^{n} \pi_{ij}u'(y)\hat{m}^j + \beta\pi_i u'(y-\gamma)m_\gamma \\ = & -(\hat{m}^i)^2 v''(\hat{m}^i) + \beta \sum_{j=1, j\neq i}^{n} \pi_{ij}u'(y)\hat{m}^j + \beta\pi_i u'(y-\gamma)m_\gamma \end{aligned}$$

are positive. The off-diagonal elements of the same matrix are negative, and the absolute sum of the off-diagonal elements is

$$\beta \sum_{j=1, j\neq i}^{n} \pi_{ij}u'(y)\hat{m}^j,$$

which is smaller than the diagonal element. Thus, the first matrix is a dominant diagonal matrix in the sense of McKenzie (1959, p.48). By the Dominant Diagonals Theorem (Theorem 2, p.49), all its characteristic roots have positive real parts. Hence, $det(-K)$, which equals the product of all its characteristic roots, is positive for every fixed point. This proves the uniqueness of the solution to (9).

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